

Representation theory of the affine Lie superalgebra $\hat{sl}(2/1; \mathbf{C})$ at fractional level.

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Abstract

$N=2$ noncritical strings are closely related to the $SL(2/1; \mathbf{R})/SL(2/1; \mathbf{R})$ Wess-Zumino-Novikov-Witten model, and there is much hope to further probe the former by using the algebraic apparatus provided by the latter. An important ingredient is the precise knowledge of the $\hat{sl}(2/1; \mathbf{C})$ representation theory at fractional level. In this paper, the embedding diagrams of singular vectors appearing in $\hat{sl}(2/1; \mathbf{C})$ Verma modules for fractional values of the level ($k = p/q - 1$, p and q coprime) are derived analytically. The nilpotency of the fermionic generators in $\hat{sl}(2/1; \mathbf{C})$ requires the introduction of a nontrivial generalisation of the MFF construction to relate singular vectors among themselves. The diagrams reveal a striking similarity with the degenerate representations of the $N = 2$ superconformal algebra.

1 Introduction

The $N = 2$ noncritical string possesses interesting features and technical challenges. In particular, as emphasized in [14], this string theory is not confined to the regime of weak gravity, i.e. the phase transition point between weak and strong gravity regimes is not of the same nature as in the $N = 0, 1$ cases. This absence of barrier in the central charge is source of complications, but also the hope of some new physics. The focus on noncritical strings in recent years was initially motivated by the nonperturbative definition of string theory in space-time dimension $d < 1$ in the context of matrix models. The continuum approach, which involves the quantisation of the Liouville theory, gives results which are in agreement with those obtained in matrix models, on the scaling behaviour of correlators for instance [21]. Although less powerful, the continuum approach generalises to supersymmetric strings. Some effort has been put in the study of $N = 1$ and $N = 2$ noncritical superstrings, but no clear picture has emerged so far as how useful they might be, in particular in extracting nonperturbative information [14, 19, 1, 5, 2].

A particularly promising tool in the description and understanding of noncritical (super)strings is the use of gauged G/G Wess-Zumino-Novikov-Witten (WZNW) models, with G a Lie (super)group, [3, 22, 18, 19]. For instance, the $SL(2/1; \mathbf{R})/SL(2/1; \mathbf{R})$ topological quantum field theory obtained by gauging the anomaly free diagonal subgroup $SL(2/1; \mathbf{R})$ of the global $SL(2/1; \mathbf{R})_L \times SL(2/1; \mathbf{R})_R$ symmetry of the WZNW model appears to be intimately related to the noncritical charged fermionic string, which is the prototype of $N = 2$ supergravity in two dimensions. A comparison of the ghost content of the two theories strongly suggests that the $N = 2$ noncritical string is equivalent to the tensor product of a *twisted* $SL(2/1; \mathbf{R})/SL(2/1; \mathbf{R})$ WZNW model with the topological theory of a spin 1/2 system [9]. It is however only when a one-to-one correspondence between the physical states and equivalence of the correlation functions of the two theories are established that one can view the twisted G/G model as the topological version of the corresponding noncritical string theory. For the bosonic string, the recent derivation of conformal blocks for admissible representations of $sl(2; \mathbf{R})$ is a major step in this direction [34].

Our aim in this paper is to provide the algebraic background for the analysis of the $N = 2$ noncritical string viewed as the $SL(2/1; \mathbf{R})/SL(2/1; \mathbf{R})$ WZNW theory. The physical states of the latter theory will be obtained in a forthcoming publication as elements of the cohomology of the BRST charge [9]. The procedure we follow is by now quite standard [8, 3, 18, 19]. The partition function of the $SL(2/1; \mathbf{R})/SL(2/1; \mathbf{R})$ theory splits in three sectors : a level k and a level $-(k+2)$ WZNW models based on $SL(2/1; \mathbf{R})$ as well as a system of four fermionic ghosts (b_a, c^a) , $a = \pm, 3, 4$ and four bosonic ghosts $(\beta_\alpha, \gamma^\alpha), (\beta'_\alpha, \gamma'_\alpha)$, $\alpha = \pm \frac{1}{2}$ corresponding to the four even (resp. odd) generators of $SL(2/1; \mathbf{R})$ [35, 29, 20, 3].

The cohomology is calculated on the space $\mathcal{F}_k \otimes \mathcal{F}_{-(k+2)} \otimes \mathcal{F}_2$ where \mathcal{F}_k denotes the space of irreducible representations of $\hat{sl}(2/1; \mathbf{R})_k$, while $\mathcal{F}_{-(k+2)}$ and \mathcal{F}_2 denote the Fock spaces of the level $-(k+2)$ and ghosts sectors respectively. As a first step, one calculates the cohomology on the whole Fock space, using a free field representation of $sl(2/1; \mathbf{R})$ and its dual. These are the Wakimoto modules constructed in [10]. In a second step, one must pass from the cohomology on the Fock space to the irreducible representations of $\hat{sl}(2/1; \mathbf{C})$ at fractional level k . After we review some basic properties of $sl(2/1; \mathbf{R})$ in Section 2, we give in Section 3 the $\hat{sl}(2/1; \mathbf{C})$ generalisation of the Malikov-Feigin-Fuchs construction needed to relate the bosonic and fermionic singular vectors whose quantum numbers are derived from the generalised Kac-Kazhdan determinant formula [28]. We stress here that the superalgebra $sl(2/1; \mathbf{R})$ has nilpotent fermionic generators, or, in other words, has lightlike fermionic roots. This property is the source of several interesting algebraic complications in the study of the representation theory of its affine counterpart, in contrast for instance with the well-studied -and much simpler case- of $osp(1/2)$, relevant for $N = 1$ noncritical strings [26, 18, 19]. Section 4 provides a classification of embedding diagrams for singular vectors appearing in highest weight Verma modules of $\hat{sl}(2/1; \mathbf{C})$ at fractional level $k = p/q - 1$, for p and q coprime. The four classes are determined by the highest weight states quantum numbers, given by the zeros of the generalised Kac-Kazhdan determinant. These diagrams are characterized by the fact that they contain an infinite number of singular vectors,

and, according to the general theory of Kac and Wakimoto [26], admissible representations should be a subset of the irreducible representations obtained as cosets of the Verma modules by the singular modules. Admissible representations, although generically non integrable, have characters which transform as finite representations of the modular group. They are ultimately the representations needed to derive the space of physical states of the $SL(2/1; \mathbf{R})/SL(2/1; \mathbf{R})$ WZNW model.

2 The Lie superalgebra $sl(2/1; \mathbf{R})$: a brief review

The set \mathcal{M} of 3×3 matrices with real entries m_{ij} whose diagonal elements satisfy the super-tracelessness condition

$$m_{11} + m_{22} - m_{33} = 0 \quad (2.1)$$

forms, with the standard laws of matrix addition and multiplication, the real Lie superalgebra $sl(2/1; \mathbf{R})$. Any matrix $\mathbf{m} \in \mathcal{M}$ can be expressed as a real linear combination of eight basis matrices

$$\begin{aligned} \mathbf{m} = & m_{11}\mathbf{h}_1 + m_{22}\mathbf{h}_2 + m_{12}\mathbf{e}_{\alpha_1+\alpha_2} + m_{21}\mathbf{e}_{-(\alpha_1+\alpha_2)} \\ & + m_{32}\mathbf{e}_{\alpha_1} + m_{23}\mathbf{e}_{-\alpha_1} + m_{13}\mathbf{e}_{\alpha_2} + m_{31}\mathbf{e}_{-\alpha_2} \end{aligned} \quad (2.2)$$

with

$$\begin{aligned} \mathbf{h}_1 &= \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right), & \mathbf{h}_2 &= \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), \\ \mathbf{e}_{\alpha_1+\alpha_2} &= \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), & \mathbf{e}_{-(\alpha_1+\alpha_2)} &= \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \\ \mathbf{e}_{\alpha_1} &= \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right), & \mathbf{e}_{-\alpha_1} &= \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), \\ \mathbf{e}_{\alpha_2} &= \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), & \mathbf{e}_{-\alpha_2} &= \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right). \end{aligned} \quad (2.3)$$

One can associate a Z_2 grading to these basis matrices by partitioning them into four submatrices of dimensions $2 \times 2, 2 \times 1, 1 \times 2$ and 1×1 , and calling even (resp. odd) those with zero off-diagonal (resp. diagonal) submatrices. From this fundamental 3-dimensional representation of $sl(2/1; \mathbf{R})$, one can write down the (anti)-commutation relations obeyed by its four bosonic generators $H_{\pm}, E_{\pm(\alpha_1+\alpha_2)}$ (corresponding to the even basis matrices $\mathbf{h}_{\pm} = \mathbf{h}_1 \pm \mathbf{h}_2, \mathbf{e}_{\pm(\alpha_1+\alpha_2)}$) and its four fermionic generators $E_{\pm\alpha_1}, E_{\pm\alpha_2}$ (corresponding to the odd basis matrices),

$$\begin{aligned} [E_{\alpha_1+\alpha_2}, E_{-(\alpha_1+\alpha_2)}] &= H_1 - H_2, & [H_1 - H_2, E_{\pm(\alpha_1+\alpha_2)}] &= \pm 2E_{\pm(\alpha_1+\alpha_2)}, \\ [E_{\pm(\alpha_1+\alpha_2)}, E_{\mp\alpha_1}] &= \pm E_{\pm\alpha_2}, & [E_{\pm(\alpha_1+\alpha_2)}, E_{\mp\alpha_2}] &= \mp E_{\pm\alpha_1}, \\ [H_1 - H_2, E_{\pm\alpha_1}] &= \pm E_{\pm\alpha_1}, & [H_1 - H_2, E_{\pm\alpha_2}] &= \pm E_{\pm\alpha_2}, \\ [H_1 + H_2, E_{\pm\alpha_1}] &= \pm E_{\pm\alpha_1}, & [H_1 + H_2, E_{\pm\alpha_2}] &= \mp E_{\pm\alpha_2}, \\ \{E_{\alpha_1}, E_{-\alpha_1}\} &= H_2, & \{E_{\alpha_2}, E_{-\alpha_2}\} &= H_1, & \{E_{\pm\alpha_1}, E_{\pm\alpha_2}\} &= E_{\pm(\alpha_1+\alpha_2)}. \end{aligned} \quad (2.4)$$

As can be seen from the above commutation relations, the even subalgebra of $sl(2/1; \mathbf{R})$ is the direct sum of an abelian algebra generated by $H_+ = H_1 + H_2$ and of the real Lie algebra $sl(2; \mathbf{R})$ generated by

$E_{\pm(\alpha_1+\alpha_2)}$ and $H_- = H_1 - H_2$. Because the semi-simple part $sl(2; \mathbf{R})$ of its even subalgebra is noncompact, the Lie superalgebra $sl(2/1; \mathbf{R})$ is a noncompact form of its complexification $A(1, 0)$. We follow here the notations of Kac [24]. According to Parker [32], the three real forms of $A(1, 0)$ are $sl(2/1; \mathbf{R})$, $su(1, 1/1)$ and $su(2/1)$. Although the latter is actually the compact form of $A(1, 0)$, the corresponding Lie supergroup is noncompact [11]. The finite dimensional irreducible representations of $sl(2/1; \mathbf{R})$, which incidentally is isomorphic to $osp(2/2; \mathbf{R})$, are constructed in [37] and in [13]. A construction of oscillator-like unitary irreducible representations of $sl(2/1; \mathbf{R})$ is given in [6]. The Cartan-Killing metric is given by a quadratic expression in the structure constants. It generalises the purely bosonic case in incorporating the Z_2 grading by associating degree zero to the bosonic generators, and degree 1 to the fermionic ones,

$$\begin{aligned} g_{\alpha\beta} &= f_{\alpha\gamma}{}^\rho f_{\beta\rho}{}^\gamma (-1)^{d(\rho)} \\ d(\rho) &= 0 \quad \text{for } \rho \text{ a bosonic index} \\ d(\rho) &= 1 \quad \text{for } \rho \text{ a fermionic index.} \end{aligned} \quad (2.5)$$

The bosonic indices take the values $\pm, \pm(\alpha_1 + \alpha_2)$, while the fermionic indices take the values $\pm\alpha_1, \pm\alpha_2$. One explicitly has

$$\begin{aligned} g_{--} &= -g_{++} = 1 \quad , \quad g_{\alpha_1+\alpha_2, -(\alpha_1+\alpha_2)} = g_{-(\alpha_1+\alpha_2), \alpha_1+\alpha_2} = 2 \\ g_{\alpha_1, -\alpha_1} &= -g_{-\alpha_1, \alpha_1} = -2 \quad , \quad g_{\alpha_2, -\alpha_2} = -g_{-\alpha_2, \alpha_2} = 2. \end{aligned} \quad (2.6)$$

The quadratic Casimir is given by,

$$\begin{aligned} C^{(2)} &= \frac{1}{2}((H_- - H_+)(H_- + H_+) + E_{\alpha_1+\alpha_2}E_{-(\alpha_1+\alpha_2)} + E_{-(\alpha_1+\alpha_2)}E_{\alpha_1+\alpha_2} \\ &\quad + E_{\alpha_1}E_{-\alpha_1} - E_{-\alpha_1}E_{\alpha_1} - E_{\alpha_2}E_{-\alpha_2} + E_{-\alpha_2}E_{\alpha_2}), \end{aligned} \quad (2.7)$$

and the atypical representations are those for which the quadratic Casimir vanishes. The fermionic nonzero roots $\pm\alpha_1, \pm\alpha_2$ have length square zero, and we normalise the bosonic nonzero roots $\pm(\alpha_1 + \alpha_2)$ to have length square 2. The root diagram can be represented in a 2-dimensional Minkowski space with the fermionic roots in the lightlike directions

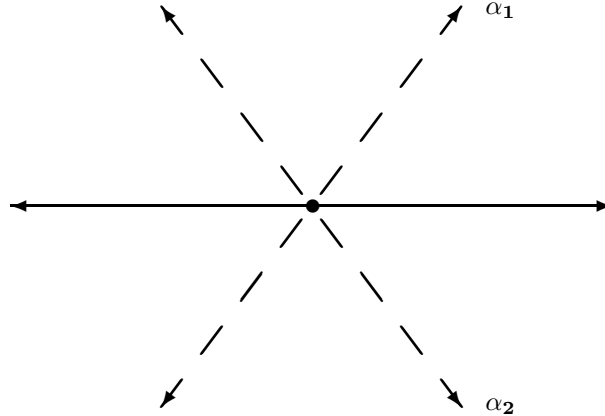


Fig.1: The root diagram of $A(1, 0)$

The Weyl group of $sl(2/1; \mathbf{R})$ is isomorphic to the Weyl group of its even simple subalgebra $sl(2; \mathbf{R})$. There is no obvious concept of a Weyl reflection about the hyperplane orthogonal to a zero square norm

fermionic root. If one therefore chooses a purely fermionic system of simple roots $\{\alpha_1, \alpha_2\}$, there is no element of the Weyl group which can transform it into the system of simple roots $\{-\alpha_2, \alpha_1 + \alpha_2\}$. Dobrev and Petkova [15] and later, Penkov and Serganova [33] have actually extended the definition of the Weyl group to incorporate the transformation $\alpha_2 \rightarrow -\alpha_2$. This non uniqueness of the generalized Dynkin diagram for Lie superalgebras is well established [25].

3 Generalisation of the Malikov-Feigin-Fuchs construction

As pointed out in the introduction, the physical spectrum of the $SL(2/1; \mathbf{R})/SL(2/1; \mathbf{R})$ WZNW topological model is determined by the structure of the Wakimoto modules given in [10], and by the structure of irreducible representations of the affine superalgebra $A(1, 0)^{(1)} \equiv \hat{sl}(2/1; \mathbf{C})$ at fractional level k .

As first discussed in [26, 25], integrable highest weight state representations of $A(1, 0)^{(1)}$ require the level k to be integer. The corresponding Verma modules, when reducible, contain an infinite number of singular vectors, and the characters of the associated irreducible representations are asserted to form a finite dimensional representation of the modular group in [26]. If one relaxes the condition $k \in \mathbf{Z}_+$ and allows the level to be fractional, the reducible Verma modules still contain an infinite number of singular vectors and, for appropriate choices of highest weight state quantum numbers, the corresponding irreducible representations still have characters written in terms of Theta functions and are believed to transform as finite dimensional representations of the modular group. Such irreducible representations are called *admissible*, according to the terminology introduced by Kac and Wakimoto [26], and they are precisely the irreducible representations which enter in the analysis of the BRST cohomology of the $SL(2/1; \mathbf{R})/SL(2/1; \mathbf{R})$ WZNW model. In order to understand their structure, we use the determinant formula for the contravariant bilinear form associated to infinite dimensional contragredient Lie superalgebras. This formula is a straight generalisation of the Kac-Kazhdan formula giving the determinant of the bilinear form associated to affine algebras [27], and appears in [28] and [16]. A more recent derivation is due to [18]. It reads,

$$\det F_\eta(\Lambda) = \prod_{n \in \mathbf{Z}_+ \setminus \{0\}} \prod_{\alpha \in \tilde{\Delta}_0^+} [\tilde{\phi}_n^{(0)}(\alpha)]^{P(\eta - n\alpha)} \prod_{n \in 1+2\mathbf{Z}_+} \prod_{\alpha \in \Delta_1^+ \setminus \tilde{\Delta}_1^+} [\phi_n^{(1)}(\alpha)]^{P(\eta - n\alpha)} \prod_{\alpha \in \tilde{\Delta}_1^+} [\tilde{\phi}^{(1)}(\alpha)]^{P_\alpha(\eta - \alpha)}, \quad (3.8)$$

with

$$\begin{aligned} \tilde{\phi}_n^{(0)}(\alpha) &= \phi_n^{(1)}(\alpha) = (\Lambda + \rho, \alpha) - \frac{1}{2}n(\alpha, \alpha) \\ \tilde{\phi}^{(1)}(\alpha) &= (\Lambda + \rho, \alpha). \end{aligned} \quad (3.9)$$

We denote by Δ the full set of roots, Δ_ϵ^+ (Δ_ϵ^-) is the set of positive (negative) even ($\epsilon = 0$) and odd ($\epsilon = 1$) roots. Also,

$$\begin{aligned} \tilde{\Delta}_0^+ &= \{\alpha \in \Delta_0^+ : \frac{1}{2}\alpha \notin \Delta\} \\ \tilde{\Delta}_1^+ &= \{\alpha \in \Delta_1^+ : (\alpha, \alpha) = 0\}. \end{aligned} \quad (3.10)$$

Determinant formulas of the kind above provide powerful information on the reducibility of Verma modules with highest weight Λ . At fixed Λ , there exists an infinity of such formulas, each corresponding to an element η of Γ_+ , the semi-group generated by the positive roots,

$$\eta = \sum_{\alpha_i \in \Delta^+} n_i \alpha_i, \quad (3.11)$$

where $n_i \in \{0, 1\}$ if $2\alpha_i \in \Delta_0^+$, and $n_i \in \mathbf{Z}_+$ otherwise. The vector ρ is defined as

$$\rho = (\bar{\rho}, h^\nu, 0) \quad (3.12)$$

where h^ν is the dual Coxeter number of the corresponding finite dimensional Lie superalgebra and $\bar{\rho}$ is half the graded sum of its positive roots,

$$\bar{\rho} = \frac{1}{2} \left(\sum_{\bar{\alpha} \in \bar{\Delta}_0^+} \bar{\alpha} - \sum_{\bar{\alpha} \in \bar{\Delta}_1^+} \bar{\alpha} \right). \quad (3.13)$$

$P(\eta)$ is the number of partitions of η , i.e. the number of ways η can be written as a linear combination of positive roots with the restrictions described in (3.11). One sets $P(0) = 1$ and $P(\eta) = 0$ if $\eta \notin \Gamma_+$. Furthermore, $P_\alpha(\eta)$ denotes the number of partitions of η which do not contain α . A criterion for irreducibility of Verma modules with highest weight Λ , $V(\Lambda)$, is that $\det F_\eta(\Lambda) \neq 0 \quad \forall \eta \in \Gamma_+$. If $\det F_\eta(\Lambda) = 0$, the Verma module contains singular vectors which generate irreducible submodules.

Let us now specialise to $A(1, 0)^{(1)}$, and extract from the Kac determinant the quantum numbers of representations of fractional level. We introduce the following Laurent expansions for the $A(1, 0)^{(1)}$ currents,

$$\begin{aligned} J(\mathbf{e}_{\pm(\alpha_1 + \alpha_2)})(z) &= \sum_n J_n^\pm z^{-n-1}, \\ J(\mathbf{h}_-)(z) &= 2 \sum_n J_n^3 z^{-n-1}, & J(\mathbf{h}_+)(z) &= 2 \sum_n U_n z^{-n-1}, \\ J(\mathbf{e}_{\pm\alpha_1})(z) &= \sum_n j_n^\pm z^{-n-1}, & J(\mathbf{e}_{\pm\alpha_2})(z) &= \sum_n j_n^\pm z^{-n-1}. \end{aligned} \quad (3.14)$$

In terms of these Laurent modes, the commutation relations for $A(1, 0)^{(1)}$ are,

$$\begin{aligned} [J_m^+, J_n^-] &= 2J_{m+n}^3 + km\delta_{m+n,0} \\ [J_m^3, J_n^\pm] &= \pm J_{m+n}^\pm, & [J_m^3, J_n^3] &= \frac{k}{2}m\delta_{m+n,0} \\ [J_m^\pm, j_n^{\mp}] &= \pm j_{m+n}^\pm, & [J_m^\pm, j_n^\mp] &= \mp j_{m+n}^{\pm'} \\ [2J_m^3, j_n^{\pm}] &= \pm j_{m+n}^{\pm'}, & [2J_m^3, j_n^\pm] &= \pm j_{m+n}^\pm \\ [2U_m, j_n^{\pm}] &= \pm j_{m+n}^{\pm'}, & [2U_m, j_n^\pm] &= \mp j_{m+n}^\pm \\ [U_m, U_n] &= -\frac{k}{2}m\delta_{m+n,0} \\ \{j_m^{\pm'}, j_n^{\mp}\} &= (U_{m+n} - J_{m+n}^3) - km\delta_{m+n,0} \\ \{j_m^+, j_n^-\} &= (U_{m+n} + J_{m+n}^3) + km\delta_{m+n,0} \\ \{j_m^{\pm'}, j_n^\pm\} &= J_{m+n}^\pm. \end{aligned} \quad (3.15)$$

The Sugawara energy-momentum tensor is given by,

$$\begin{aligned} T(z) &= \frac{1}{2(k+1)} \{ 2(J^3)^2(z) - 2U^2(z) + J^+ J^-(z) + J^- J^+(z) \\ &\quad + j^{\pm'} j^{\mp}(z) - j^{\mp'} j^{\pm}(z) - j^+ j^-(z) + j^- j^+(z) \}. \end{aligned} \quad (3.16)$$

Its zero-mode subalgebra possesses an automorphism τ of order 2,

$$\begin{aligned} \tau(J^\pm) &= J^\pm, & \tau(J^3) &= J^3, & \tau(U) &= U \\ \tau(j^\pm) &= -j^\pm, & \tau(j^{\pm'}) &= -j^{\pm'}, \end{aligned} \quad (3.17)$$

which can be used to introduce the following twist in the affine superalgebra $A(1,0)^{(1)}$,

$$\begin{aligned} (J^\pm)'_n &= J_{n\pm 1}^\pm, & (J^3)'_n &= J_n^3 + \frac{k}{2}\delta_{n,0}, & U'_n &= U_n \\ (j^\pm)'_n &= j_{n\pm\frac{1}{2}}^\pm, & (j')^\pm'_n &= j_{n\pm\frac{1}{2}}^{\pm'}. \end{aligned} \quad (3.18)$$

Let us extend the superalgebra (3.15) by L_0 , the zero-mode operator in the Laurent expansion of $T(z)$. It is straightforward to check that the commutation relations (3.15), together with the extra relations,

$$[L_0, \phi_n] = -n\phi_n, \quad \phi_n = J_n^\pm, J_n^3, U_n, j_n^\pm, j_n^{\pm'}, \quad (3.19)$$

are unchanged when one considers the primed operators (3.18) and

$$L'_0 = L_0 + J_0^3. \quad (3.20)$$

The unprimed superalgebra (3.15) with $m, n \in \mathbf{Z}$ is called the Ramond sector of the theory, while the primed twisted superalgebra is known as the Neveu-Schwarz sector. The above discussion shows that the two sectors are isomorphic. The conformal weight, isospin and $U(1)$ charges of physical states are related in the following way between the two sectors,

$$h^{NS} = h^R + \frac{1}{2}h_-^R, \quad \frac{1}{2}h_-^{NS} = \frac{1}{2}h_-^R + \frac{k}{2}, \quad \frac{1}{2}h_+^{NS} = \frac{1}{2}h_+^R. \quad (3.21)$$

For definiteness in this paper, all our subsequent discussions are in the Ramond sector, and we choose the two simple roots to be fermionic (Type I). The $A(1,0)^{(1)}$ root lattice is generated, in the type I, Ramond picture, by three simple roots,

$$\begin{aligned} \alpha_0 &= (-(\bar{\alpha}_1 + \bar{\alpha}_2), 0, 1) \\ \alpha_i &= (\bar{\alpha}_i, 0, 0), \quad i = 1, 2 \end{aligned} \quad (3.22)$$

where $\{\bar{\alpha}_1, \bar{\alpha}_2\}$ are the two fermionic roots of $A(1,0)^{(1)}$ introduced in Section 2. The set of positive roots, Δ^+ , can be written as

$$\Delta^+ = (\Delta_0^+ \setminus \tilde{\Delta}_0^+) \cup \tilde{\Delta}_0^+ \cup (\Delta_1^+ \setminus \tilde{\Delta}_1^+) \cup \tilde{\Delta}_1^+, \quad (3.23)$$

where, in the case of $A(1,0)^{(1)}$, $\tilde{\Delta}_0^+ = \Delta_0^+ \setminus \{(0, 0, 2m)\}$ and $\tilde{\Delta}_1^+ = \tilde{\Delta}_1^+$. One has,

$$\begin{aligned} \tilde{\Delta}_0^+ &= \{(\bar{\alpha}_1 + \bar{\alpha}_2), 0, m), (-(\bar{\alpha}_1 + \bar{\alpha}_2), 0, 1 + m), (0, 0, 1 + 2m), m \in \mathbf{Z}_+\}, \\ \tilde{\Delta}_1^+ &= \{(\bar{\alpha}_i, 0, m), (-\bar{\alpha}_i, 0, 1 + m), m \in \mathbf{Z}_+, i = 1, 2\}, \end{aligned} \quad (3.24)$$

and therefore,

$$\rho = (0, h^\nu, 0). \quad (3.25)$$

The dual Coxeter number of $A(1,0)^{(1)}$ is independent of the choice of simple roots [25] and is $h^\nu = 1$.

Let us parametrise the highest weight vector by the two quantum numbers h_\pm , corresponding to the eigenvalues of the Cartan operators $H_\pm = H_1 \pm H_2$ (2.4), and by the level k at which the affine algebra $A(1,0)^{(1)}$ is considered,

$$\Lambda = (\bar{\Lambda}, k, 0) = \left(\frac{1}{2}h_- (\bar{\alpha}_1 + \bar{\alpha}_2) + \frac{1}{2}h_+ (\bar{\alpha}_1 - \bar{\alpha}_2), k, 0\right) \quad (3.26)$$

(note that the notion of fundamental weight is ill-defined whenever a simple root has zero length). The different factors in the determinant formula are (recall $(\bar{\alpha}_1 + \bar{\alpha}_2)^2 = 2$, $m \in \mathbf{Z}_+$) and $n \in \mathbf{Z}_+ \setminus \{0\}$,

$$\begin{aligned} \tilde{\phi}_n^{(0)}((\bar{\alpha}_1 + \bar{\alpha}_2), 0, m) &= h_- + (k+1)m - n \\ \tilde{\phi}_n^{(0)}((-\bar{\alpha}_1 + \bar{\alpha}_2), 0, 1 + m) &= -h_- + (k+1)(1+m) - n \end{aligned}$$

$$\begin{aligned}
\tilde{\phi}^{(0)}((0, 0, 1 + 2m)) &= (k + 1)(1 + 2m) \\
\tilde{\phi}^{(1)}((\bar{\alpha}_i, 0, m)) &= \frac{1}{2}h_- + (-1)^i \frac{1}{2}h_+ + (k + 1)m, \quad i = 1, 2 \\
\tilde{\phi}^{(1)}((-\bar{\alpha}_i, 0, 1 + m)) &= -\frac{1}{2}h_- - (-1)^i \frac{1}{2}h_+ + (k + 1)(1 + m), \quad i = 1, 2.
\end{aligned} \tag{3.27}$$

Our aim is to provide the embedding diagrams and quantum numbers of singular vectors within Verma modules built on highest weights Λ (3.26) whose quantum numbers h_{\pm}, k lie at the intersection of infinitely many lines $\tilde{\phi}_n^{(0)}(\alpha) = \tilde{\phi}^{(1)}(\alpha') = 0$. This happens when

$$k + 1 = p/q, \quad \gcd(p, q) = 1, \tag{3.28}$$

since one has then,

$$\begin{aligned}
\tilde{\phi}_n^{(0)}(((\bar{\alpha}_1 + \bar{\alpha}_2), 0, m)) &= \tilde{\phi}_{n+\nu p}^{(0)}(((\bar{\alpha}_1 + \bar{\alpha}_2), 0, m + \nu q)) \\
\tilde{\phi}_n^{(0)}((-(\bar{\alpha}_1 + \bar{\alpha}_2), 0, 1 + m)) &= \tilde{\phi}_{n+\nu p}^{(0)}((-(\bar{\alpha}_1 + \bar{\alpha}_2), 0, 1 + m + \nu q))
\end{aligned} \tag{3.29}$$

for $\nu \in \mathbf{Z}$.

Note that for an irreducible highest weight module over $A(1, 0)^{(1)}$ to be integrable, the conditions on

$$m_i = (\Lambda, \alpha_i), \quad i = 0, 1, 2 \tag{3.30}$$

are [25]

$$\begin{aligned}
m_1 + m_2 &= h_- \in \mathbf{Z}_+ \setminus \{0\} \quad \text{or} \quad m_1 = m_2 = 0, \text{ i.e. } h_- = h_+ = 0 \\
m_0 &= k - h_- \in \mathbf{Z}_+,
\end{aligned} \tag{3.31}$$

which corresponds in (3.28), to considering $q = 1$.

In order to construct the embedding diagrams, we first encode the information on singular vectors provided by the zeros of the Kac-Kazhdan determinant in the following definitions and lemmas. We restrict our analysis to the case $k + 1 \neq 0$.

Definition 1 : A singular vector χ of an $A(1, 0)^{(1)}$ Verma module is a zero norm vector such that $J_1^- \chi = j_0^+ \chi = j_0^{+'} \chi = 0$ (in the Ramond sector). A highest weight state is a singular vector whose square length is strictly positive.

Definition 2 : A subsingular vector Σ of an $A(1, 0)^{(1)}$ Verma module is a vector such that the three vectors $J_1^- \Sigma, j_0^+ \Sigma, j_0^{+'} \Sigma$ but not Σ itself can be made to vanish by setting at least one singular vector to zero in the Verma module. A more mathematically precise definition can be found in [12]. Subsingular vectors are not given by the Kac-Kazhdan formula and are not included in our diagrams.

Lemma 1 : a) If χ is a singular vector such that $L_0 \chi = H \chi$, $J_0^3 \chi = \frac{1}{2} H_- \chi$ and $U_0 \chi = \frac{1}{2} H_+ \chi$, and if $H_- + (k + 1)m - n = 0$ for some $m \in \mathbf{Z}_+$ and $n \in \mathbf{Z}_+ \setminus \{0\}$, there exists a singular vector corresponding to $\eta = n((\bar{\alpha}_1 + \bar{\alpha}_2), 0, m)$ with conformal weight $H + mn$, isospin $\frac{1}{2} H_- - n$ and charge $\frac{1}{2} H_+$.

b) If χ is a singular vector such that $L_0 \chi = H \chi$, $J_0^3 \chi = \frac{1}{2} H_- \chi$ and $U_0 \chi = \frac{1}{2} H_+ \chi$, and if $H_- - (k + 1)(1 + m) + n = 0$ for some $m \in \mathbf{Z}_+$ and $n \in \mathbf{Z}_+ \setminus \{0\}$, there exists a singular vector corresponding to $\eta = n(-(\bar{\alpha}_1 + \bar{\alpha}_2), 0, 1 + m)$ with conformal weight $H + (1 + m)n$, isospin $\frac{1}{2} H_- + n$ and charge $\frac{1}{2} H_+$.

Lemma 1 allows one to obtain all the uncharged descendants of any singular vector in an iterative way, by using modified affine Weyl reflexions w_0, w_1 of the $\widehat{SU}(2)$ subalgebra of $A(1,0)^{(1)}$. Indeed, if χ is a singular vector with quantum numbers H, H_- and H_+ , then

(1) the vector $w_0\chi = [J_{-1}^+]^{k+1-H_-}\chi$ is singular in the Ramond sector for $k+1-H_-$ positive integer

(2) the vector

$$w_1\chi = [J_0^-]^{H_- - 1}(2H_- j_0^- j_0^{-'} + [H_+ - H_-]J_0^-)\chi = 2\{j_0^+, (J_0^-)^{H_-} j_0^-\}\chi = 2\{j_0^{+'}, (J_0^-)^{H_-} j_0^{-'}\}\chi$$

is singular in the Ramond sector for $H_- - 1$ positive integer.

Whenever the powers $k+1-H_-$ and $H_- - 1$ are not positive integers, the above construction must be analytically continued a la Malikov-Feigin-Fuchs. In order to describe this construction in the specific case of $\hat{sl}(2/1; \mathbf{C})$, let us introduce the vector,

$$\begin{aligned} \tilde{w}_0^{(M)}\chi = \\ \prod_{i=1}^M \left\{ (J_0^-)^{2i(k+1)-H_- - 1} ([-2H_- + 4i(k+1)] j_0^- j_0^{-'} + [H_+ + H_- - 2i(k+1)] J_0^-) \right. \\ \left. \times (J_{-1}^+)^{(2i-1)(k+1)-H_-} \right\} \chi \end{aligned} \quad (3.32)$$

with quantum numbers

$$\begin{aligned} H' &= H + M^2(k+1) - MH_- \\ H'_- &= H_- - 2M(k+1) \\ H'_+ &= H_+, \end{aligned} \quad (3.33)$$

and the vector

$$\begin{aligned} \tilde{w}_1^{(M)}\chi = \\ \prod_{i=0}^{M-1} \left\{ (J_{-1}^+)^{(2i+1)(k+1)+H_-} \right. \\ \left. \times (J_0^-)^{2i(k+1)+H_- - 1} ([2H_- + 4i(k+1)] j_0^- j_0^{-'} + [H_+ - H_- - 2i(k+1)] J_0^-) \right\} \chi \end{aligned} \quad (3.34)$$

with quantum numbers

$$\begin{aligned} H' &= H + M^2(k+1) + MH_- \\ H'_- &= H_- + 2M(k+1) \\ H'_+ &= H_+. \end{aligned} \quad (3.35)$$

We also define the vector

$$w_0^{(M)}\chi = (J_{-1}^+)^{(2M+1)(k+1)-H_-} \tilde{w}_0^{(M)}\chi \quad (3.36)$$

with quantum numbers

$$\begin{aligned} H' &= H + (M+1)^2(k+1) - (M+1)H_- \\ H'_- &= -H_- + 2(M+1)(k+1) \\ H'_+ &= H_+, \end{aligned} \quad (3.37)$$

and the vector

$$\begin{aligned} w_1^{(M)} \chi &= \\ (J_0^-)^{2M(k+1)+H_- -1} ([2H_- + 4M(k+1)] j_0^- j_0^{-'} + [H_+ - H_- - 2M(k+1)] J_0^-)] \tilde{w}_1^{(M)} \chi \end{aligned} \quad (3.38)$$

with quantum numbers

$$\begin{aligned} H' &= H + M^2(k+1) + MH_- \\ H'_- &= -H_- - 2M(k+1) \\ H'_+ &= H_+. \end{aligned} \quad (3.39)$$

In the above expressions, M is a positive integer such that $M(k+1) \pm H_-$ is a positive integer. The products are ordered in such a way that the factor evaluated at $i+1$ is at the left of the factor evaluated at i . A remarkable property of $w_0^{(M)}$ and $w_1^{(M)}$ is that their square is either zero or proportional to the identity, namely,

$$(w_0^{(M)})^2 \chi = \prod_{i=1}^M [H_+ + H_- - 2(M+1-i)(k+1)] [H_+ - H_- + 2(M+1-i)(k+1)] \chi \quad (3.40)$$

and

$$(w_1^{(M)})^2 \chi = \prod_{i=0}^M [H_+ + H_- + 2(M-i)(k+1)] [H_+ - H_- - 2(M-i)(k+1)] \chi. \quad (3.41)$$

Similarly, one has,

$$\tilde{w}_1^{(M)} \tilde{w}_0^{(M)} \chi = \prod_{i=1}^M (H_+ + H_- - 2i(k+1)) (H_+ - H_- + 2i(k+1)) \quad (3.42)$$

and

$$\tilde{w}_0^{(M)} \tilde{w}_1^{(M)} \chi = \prod_{i=0}^{M-1} (H_+ + H_- + 2i(k+1)) (H_+ - H_- - 2i(k+1)). \quad (3.43)$$

These properties are easily derived from the definitions (3.32), (3.34), (3.37), (3.38) and the relation,

$$(J_0^-)^{-1} (\alpha j_0^{-'} j_0^- + \beta J_0^-) (J_0^-)^{-1} (\alpha j_0^- j_0^{-'} + \beta J_0^-) = \beta(\beta + \alpha), \quad (3.44)$$

for α and β c-numbers.

The analytically continued version of the singular vectors $w_0 \chi$ and $w_1 \chi$ constructed above is therefore given, when χ is a singular vector with the quantum numbers of Lemma 1, by,

$$(1) \text{ the singular vector } w_0^{(q-m-1)} \chi$$

(2) the singular vector $w_1^{(m)}\chi$.

The case where $k+1$ is integer corresponds to $q=1, m=0$ and one has $w_0^{(0)}\chi = w_0\chi, w_1^{(0)}\chi = w_1\chi$.

Lemma 2 : If χ is a singular vector such that $L_0\chi = H\chi, J_0^3\chi = \frac{1}{2}H_-\chi$ and $U_0\chi = \frac{1}{2}H_+\chi$, and if $H_+ - H_- = 2(k+1)M$ for some integer $M \geq 0$, there exists a singular vector corresponding to $\eta = (\bar{\alpha}_1, 0, M)$ with conformal weight $H+M$, isospin $\frac{1}{2}H_- - \frac{1}{2}$ and charge $\frac{1}{2}H_+ - \frac{1}{2}$. It is given by,

$$\xi = \tilde{w}_0^{(M)} j_0^{-'} \tilde{w}_1^{(M)} \chi. \quad (3.45)$$

Note that for $M=0$, this singular vector is given by $j_0^{-'}\chi$. For $M=1$, the construction above gives

$$\begin{aligned} \xi = & -4(k+1+H_-) (k+H_-) \times \\ & \{ ((k+1)J_0^- - j_0^{-'} j_0^-) j_{-1}^+ - (k+1)H_- j_{-1}^- - j_0^{-'} [J_0^- J_{-1}^+ - H_- J_{-1}^3 + H_- U_{-1}] \} \chi. \end{aligned} \quad (3.46)$$

When $k+1+H_- = 0$, the singular vector is proportional to

$$\{ ((k+1)J_0^- - j_0^{-'} j_0^-) j_{-1}^+ + (k+1)^2 j_{-1}^- - j_0^{-'} [J_0^- J_{-1}^+ + (k+1)J_{-1}^3 - (k+1)U_{-1}] \} \chi. \quad (3.47)$$

Lemma 3 : If χ is a singular vector such that $L_0\chi = H\chi, J_0^3\chi = \frac{1}{2}H_-\chi$ and $U_0\chi = \frac{1}{2}H_+\chi$, and if $H_+ - H_- = -2(k+1)(1+M)$ for some integer $M \geq 0$, there exists a singular vector corresponding to $\eta = (-\bar{\alpha}_1, 0, 1+M)$ with conformal weight $H+1+M$, isospin $\frac{1}{2}H_- + \frac{1}{2}$ and charge $\frac{1}{2}H_+ + \frac{1}{2}$. It is given by,

$$\xi = w_0^{(M)} j_0^- w_0^{(M)} \chi. \quad (3.48)$$

Note that for $M=0$, this singular vector is $[(k-H_-)j_{-1}^{+'} + j_0^- J_{-1}^+]\chi$.

Lemma 4 : If χ is a singular vector such that $L_0\chi = H\chi, J_0^3\chi = \frac{1}{2}H_-\chi$ and $U_0\chi = \frac{1}{2}H_+\chi$, and if $H_+ + H_- = 2(k+1)(1+M)$ for some integer $M \geq 0$, there exists a singular vector corresponding to $\eta = (-\bar{\alpha}_2, 0, 1+M)$ with conformal weight $H+1+M$, isospin $\frac{1}{2}H_- + \frac{1}{2}$ and charge $\frac{1}{2}H_+ - \frac{1}{2}$. It is given by,

$$\xi = w_0^{(M)} j_0^{-'} w_0^{(M)} \chi. \quad (3.49)$$

For $M=0$, it is given by

$$[(H_- - k)j_{-1}^+ + j_0^{-'} J_{-1}^+]\chi. \quad (3.50)$$

Lemma 5 : If χ is a singular vector such that $L_0\chi = H\chi, J_0^3\chi = \frac{1}{2}H_-\chi$ and $U_0\chi = \frac{1}{2}H_+\chi$, and if $H_+ + H_- = -2(k+1)M$ for some integer $M \geq 0$, there exists a singular vector corresponding to $\eta = (\bar{\alpha}_2, 0, M)$ with conformal weight $H+M$, isospin $\frac{1}{2}H_- - \frac{1}{2}$ and charge $\frac{1}{2}H_+ + \frac{1}{2}$. It is given by,

$$\xi = \tilde{w}_0^{(M)} j_0^- \tilde{w}_1^{(M)} \chi. \quad (3.51)$$

For $M=0$, it is given by $j_0^-\chi$.

4 Embedding diagrams

The embedding diagrams we construct describe all singular vectors, given with their multiplicity, within a given Verma module with highest weight state Λ taken as bosonic for definiteness. An arrow originating at a singular vector v_1 and pointing at a singular vector v_2 expresses that v_2 is a descendant of v_1 , and that there is no singular vector v_3 such that v_2 is a descendant of v_3 , itself descendant of v_1 . We split the embedding diagrams in four classes, described below as classes I to IV, according to whether the Kac-Kazhdan determinant $\det F_\eta(\Lambda)$ for Λ highest weight vector has none (class I), one (class II), or two zeros (classes III and IV) in the fermionic sector

$$\begin{aligned}\tilde{\phi}^{(1)}((\bar{\alpha}_i, 0, m)) &= 0, & i = 1, 2 \\ \tilde{\phi}^{(1)}((-\bar{\alpha}_i, 0, 1 + m)) &= 0 & i = 1, 2,\end{aligned}\tag{4.52}$$

see (3.27). The standard technique to obtain the embedding structure of singular vectors is based on an iteration procedure. First, one considers the zeros of the determinant formula $\det F_\eta(\Lambda) = 0$ for Λ the highest weight state, and uses the five lemmas of Section 3 to draw the relevant connecting arrows between the singular vectors corresponding to these zeros. Not all singular vectors are obtained from $\det F_\eta(\Lambda) = 0$ however; one must consider the zeros of $\det F_\eta(\Lambda') = 0$ for any singular vector Λ' identified in the previous stage, and use the lemmas again. This iterative procedure will produce all singular vectors within a given Verma module. It may however fail to provide the correct embedding structure in three ways. Indeed,

- (1) it will not recognise if a singular vector vanishes identically,
- (2) it will not provide a complete set of interrelating arrows between singular vectors,
- (3) it will not give the multiplicity of each singular vector.

It is therefore extremely useful to combine it with the knowledge of analytic expressions for singular vectors in order to provide a complete embedding diagram. Let us first illustrate how analytic expressions allow one to identify vanishing singular vectors. Let Z'_0 be a bosonic highest weight vector whose quantum numbers obey the relation $h_+ - h_- = 2(k+1)M$, $M \geq 0$. Using Lemma 2, one constructs a fermionic singular vector

$$Z'^{-}_0 = \tilde{w}_0^{(M)} j'^{-}_0 \tilde{w}_1^{(M)} Z'_0 \tag{4.53}$$

whose quantum numbers satisfy the same relation $H_+ - H_- = 2(k+1)M$. If one considers $\det F_\eta(Z'^{-}_0) = 0$, Lemma 2 produces the singular vector

$$Z'^{-}_0 = \tilde{w}_0^{(M)} j'^{-}_0 \tilde{w}_1^{(M)} Z'^{-}_0. \tag{4.54}$$

However, it identically vanishes, as can be seen by replacing Z'^{-}_0 by its expression (4.53) and using the result (3.42) together with the fact that j'^{-}_0 is a nilpotent fermionic generator.

We now show how analytic expressions allow to obtain relations between singular vectors which are not in the determinant formula. Take for instance a highest weight vector Z'_0 with $h_+ = h_- \in \mathbf{Z}_+ \setminus \{0\}$. By Lemma 2, there exists a fermionic singular vector $Z'^{-}_0 = j'^{-}_0 Z'_0$ with $H_+ = H_- = h_- - 1$. By Lemma 1, there also exists a singular vector

$$\begin{aligned}T'_0 = w_1 Z'_0 &= (J_0^-)^{h_- - 1} (2h_- j_0^- j'^{-}_0) Z'_0 \\ &= 2h_- (J_0^-)^{h_- - 1} j_0^- Z'^{-}_0.\end{aligned}\tag{4.55}$$

This shows how T'_0 is a descendant of Z'^{-}_0 , a relation missed by the standard iterative procedure. We would like to stress at this point that if Z'^{-}_0 were the highest weight state of the Verma module, T'_0 would be a *subsingular* vector, in the sense of definition 2, and we would not have included it in the embedding diagram.

But T'_0 becomes a singular vector when Z'^{-}_0 is considered as a fermionic descendant of the highest weight state Z'_0 . So the missing arrows are always connected to the presence of subsingular vectors in the sense just described. Such a situation occurs in all embedding diagrams where bosonic and fermionic singular vectors coexist. We will stress it again in our discussion of class II.

Finally, the multiplicity of singular vectors is usually one, except for one particular class of highest weight vectors. We will discuss this issue in plenty details below, in the context of class IV.

In the following, we concentrate on Verma modules built on highest weight vectors Λ (see (3.26)) whose quantum number h_- obeys the constraint

$$h_- + (k+1)m - n = 0 \quad (4.56)$$

where m, n are two integers such that

$$0 \leq m \leq q-1 \quad \text{and} \quad 0 \leq n \leq p-1, \quad (4.57)$$

and

$$k+1 = p/q, \quad p, q \in \mathbf{Z}_+ \setminus \{0\}, \quad \gcd(p, q) = 1. \quad (4.58)$$

As explained above, the condition of fractional level k together with condition (4.56) are a necessary requirement for the Verma module to possess an infinite number of singular vectors. The embedding diagrams have different structures according to whether or not extra conditions are imposed on the highest weight vector quantum numbers h_{\pm} . The invariance (3.29) of equation (4.56) under the shift

$$m \rightarrow m + \nu q, \quad n \rightarrow n + \nu p \quad (4.59)$$

allows one to choose m in the range (4.57). The integer n can be parametrized as

$$n = (\rho - 1)p + \tilde{n}, \quad \rho \geq 1, \quad 0 \leq \tilde{n} \leq p-1, \quad (4.60)$$

but we restrict our analysis to the value $\rho = 1$. It is indeed for this value of ρ that the characters of the corresponding irreducible representations may be written in terms of generalised Theta functions. However, the condition $\rho = 1$ is not sufficient to characterise admissible representations. For instance, the Verma modules whose highest weight vectors quantum number h_- satisfies (4.56) when $n = 0$ (edge of the Kac table) do not lead to admissible representations.

As already mentioned above, the Verma modules considered here fall into four classes. If $|\Lambda\rangle$ is the Verma module bosonic highest weight state, with quantum numbers $h, \frac{1}{2}h_-, \frac{1}{2}h_+$ given by

$$L_0|\Lambda\rangle = h|\Lambda\rangle, \quad J_0^3|\Lambda\rangle = \frac{1}{2}h_-|\Lambda\rangle, \quad U_0|\Lambda\rangle = \frac{1}{2}h_+|\Lambda\rangle, \quad (4.61)$$

we have,

Class I : $|\Lambda\rangle$ has conformal weight $h \geq 0$, arbitrary real charge $\frac{1}{2}h_+$ and isospin $\frac{1}{2}h_-$ obeying (4.56)

$$h_- + (k+1)m - n = 0, \quad 0 \leq m \leq q-1, \quad 0 \leq n \leq p-1. \quad (4.62)$$

All singular vectors are bosonic descendants of the highest weight state, and therefore have the same fixed arbitrary real charge $\frac{1}{2}h_+$. They are organised in four families labeled by a positive integer $a \geq 0$, with quantum numbers

$$\begin{aligned} Z'_a &: H_a = h + a^2 pq + a(qn - pm), \\ &\quad (h_-)_a = \mathbf{n} + 2\mathbf{a}p - m(k+1) \\ T'_a &: H_a = h + mn + a^2 pq + a(qn + pm), \end{aligned}$$

$$\begin{aligned}
(h_-)_a &= -\mathbf{n} - 2\mathbf{a}\mathbf{p} - m(k+1) \\
Z_{a+1} &: H_{a+1} = h + mn + (a+1)^2 pq - (a+1)(qn + pm), \\
&\quad (h_-)_{a+1} = -\mathbf{n} + 2(\mathbf{a} + 1)\mathbf{p} - m(k+1) \\
T_{a+1} &: H_{a+1} = h + (a+1)^2 pq - (a+1)(qn - pm), \\
&\quad (h_-)_{a+1} = \mathbf{n} - 2(\mathbf{a} + 1)\mathbf{p} - m(k+1).
\end{aligned} \tag{4.63}$$

Note that at the edge of the Kac table, when $n = 0$, one has the following identification,

$$Z'_{a+1} \equiv Z_{a+1}, \quad \text{and} \quad T'_{a+1} \equiv T_{a+1}. \tag{4.64}$$

We refer to this case as the collapsed version of the generic case $n \neq 0$. The corresponding embedding diagrams (Figure 2a and Figure 2b) are constructed in an iterative way by using the lemmas above. One has, in terms of modified Weyl transformations,

$$\begin{aligned}
Z'_a &= (w_0^{(q-m-1)} w_1^{(m)})^a Z'_0 \\
T'_a &= (w_1^{(m)} w_0^{(q-m-1)})^a w_1^{(m)} Z'_0 \\
Z_{a+1} &= (w_0^{(q-m-1)} w_1^{(m)})^a w_0^{(q-m-1)} Z'_0 \\
T_{a+1} &= (w_1^{(m)} w_0^{(q-m-1)})^{a+1} Z'_0.
\end{aligned} \tag{4.65}$$

Class II : $|\Lambda\rangle$ has conformal weight $h \geq 0$, but the charge and isospin obey the following constraints,

$$h_- + (k+1)m - n = 0 \quad \text{and} \quad h_- - h_+ = -2(k+1)m' \tag{4.66}$$

which implies

$$h_- + h_+ = 2(k+1)(m' - m) + 2n. \tag{4.67}$$

Here,

$$0 \leq m \leq q-1, \quad 1 \leq n \leq p-1, \quad m' \in \mathbf{Z}_+ \tag{4.68}$$

and

$$m' - m = (\sigma - 1)q + \tilde{m}, \quad \sigma \in \mathbf{Z}_+, \quad 0 \leq \tilde{m} \leq q-1. \tag{4.69}$$

The case where m' is a negative integer is totally similar and leads to embedding diagrams which are mirror images of the ones presented in Figure 3 and Figure 4. The singular vectors have charge $\frac{1}{2}h_+$ when they are bosonic descendants of the highest weight state, and their quantum numbers are those of (4.63). The fermionic descendants have charge $\frac{1}{2}H_+ - \frac{1}{2}$, and their other quantum numbers are obtained from (4.63) by shifting

$$n \rightarrow n-1, \quad h \rightarrow h + m'. \tag{4.70}$$

The value $n = 1$ corresponds to a degenerate (collapsed) situation. The embedding diagrams for this case and the case $n \neq 1$ are given in Figure 3 and Figure 4 when $\tilde{m} = 0$. If $\tilde{m} \neq 0$, one must distinguish between the cases when $0 \leq m + \tilde{m} \leq q-1$ and $q \leq m + \tilde{m} \leq 2q-1$. However, the diagrams have the same structure as in the $\tilde{m} = 0$ case. The only difference is in the singular vector sitting at the annihilation node in the fermionic sector.

Unlike Class I, Class II possesses bosonic and fermionic singular vectors in the same Verma module. The nilpotency of the fermionic generators in $A(1,0)^{(1)}$ has a crucial impact on the way the singular vectors are related. In Figure 3 for instance, the singular vector T_σ is not a descendant of $T'_{\sigma-1}$: the ‘path’ between these two vectors is formally given by

$$T_\sigma = (w_1^{(m)} w_0^{(q-m-1)})^\sigma w_1^{(m)} (w_0^{(q-m-1)} w_1^{(m)})^{\sigma-1} T'_{\sigma-1}, \tag{4.71}$$

but one also has $T'_{\sigma-1} = w_1^{(m)} Z'_{\sigma-1}$. It can be shown that $w_1^{(m)} w_1^{(m)} Z'_{\sigma-1}$ is zero, using (3.41).

In order to connect singular vectors of charge h_+ to singular vectors of charge $h_+ - 1$, one uses the transformations $w_0^{(q-m-1)}$ and $w_1^{(m)}$ as well as two fundamental fermionic transformations. The first one relates $Z'_{\sigma-1}$ and $Z'^{-}_{\sigma-1}$ using Lemma 2, namely,

$$Z'^{-}_{\sigma-1} = \tilde{w}_0^{(m+\tilde{m})} j_0'^{-} \tilde{w}_1^{(m+\tilde{m})} Z'_{\sigma-1}, \quad (4.72)$$

which reduces to $Z'^{-}_{\sigma-1} = j_0'^{-} Z'_{\sigma-1}$ when $m + \tilde{m} = 0$. The second one is not of the kind given in the lemmas. Although it connects two singular vectors which correspond to zeros of the Kac determinant, the latter does not encode the fact that one is the descendant of the other. This second basic fermionic transformation relates $Z'^{-}_{\sigma-1}$ and $T'_{\sigma-1}$ as follows,

$$T'_{\sigma-1} = \tilde{w}_1^{(\tilde{m})} (J_0^-)^{h_+-1} j_0^- \tilde{w}_1^{(m+\tilde{m})} Z'^{-}_{\sigma-1}. \quad (4.73)$$

Class III : $|\Lambda\rangle$ has conformal weight $h \geq 0$, but the charge and isospin obey the following constraints,

$$h_- + (k+1)m = 0 \quad \text{and} \quad h_- - h_+ = -2(k+1)m' \quad (4.74)$$

which implies

$$h_- + h_+ = 2(k+1)(m' - m). \quad (4.75)$$

Here,

$$0 \leq m \leq q-1, \quad m' \in \mathbf{Z}_+, \quad (4.76)$$

and

$$m' - m = (\sigma-1)q + \tilde{m} \geq 1, \quad \sigma \in \mathbf{Z}_+, \quad 0 \leq \tilde{m} \leq q-1. \quad (4.77)$$

The case where m' is a negative integer produces embedding diagrams which are mirror images of the diagrams in Figure 5 and Figure 6. The bosonic singular vectors have charge $\frac{1}{2}h_+$, with the other quantum numbers given by (4.63) when $n = 0$,

$$\begin{aligned} Z'_a &: H_a = h + a^2 pq - apm, \\ &\quad (h_-)_a = \mathbf{2ap} - m(k+1) \\ T'_{a+1} &: H_{a+1} = h + (a+1)^2 pq + (a+1)pm, \\ &\quad (h_-)_{a+1} = -\mathbf{2(a+1)p} - m(k+1), \end{aligned} \quad (4.78)$$

with $a \geq 0$. The fermionic singular vectors have charge $\frac{1}{2}h_+ - \frac{1}{2}$, with quantum numbers

$$\begin{aligned} Z'^{-}_a &: H_a = h + m' - m + a^2 pq + a(q - pm), \\ &\quad (h_-)_a = \mathbf{1} + \mathbf{2ap} - m(k+1) \\ T'^{-}_a &: H_a = h + m' + a^2 pq + a(q + pm), \\ &\quad (h_-)_a = -\mathbf{1} - \mathbf{2ap} - m(k+1) \\ Z'^{-}_{a+1} &: H_{a+1} = h + m' + (a+1)^2 pq - (a+1)(q + pm), \\ &\quad (h_-)_{a+1} = -\mathbf{1} + \mathbf{2(a+1)p} - m(k+1) \\ T'^{-}_{a+1} &: H_{a+1} = h + m' - m + (a+1)^2 pq - (a+1)(q - pm), \\ &\quad (h_-)_{a+1} = \mathbf{1} - \mathbf{2(a+1)p} - m(k+1). \end{aligned} \quad (4.79)$$

The two diagrams in Figure 5 and Figure 6 correspond to the cases $p = 1$ and $p \neq 1$ respectively, with $\tilde{m} = 0$. If $\tilde{m} \neq 0$, one must, as in Class II, distinguish between the cases $0 \leq m + \tilde{m} \leq q-1$ and $q \leq m + \tilde{m} \leq 2q-1$. However, the diagrams have the same structure as the ones given here, and we omit them.

Class IV : $|\Lambda >$ has conformal weight $h \geq 0$, but the charge and isospin obey the following constraints,

$$h_- + (k+1)m = 0 \quad \text{and} \quad h_- - h_+ = -2(k+1)m' \quad (4.80)$$

which implies

$$h_- + h_+ = 2(k+1)(m' - m). \quad (4.81)$$

Here,

$$0 \leq m \leq q-1, \quad m' \in \mathbf{Z}_+, \quad (4.82)$$

but

$$m' - m = (\sigma - 1)q + \tilde{m} \leq 0, \quad \sigma \in \mathbf{Z}_+, 0 \leq \tilde{m} \leq q-1. \quad (4.83)$$

The bosonic singular vectors have charge $\frac{1}{2}h_+$, with the other quantum numbers given by (4.78). The fermionic singular vectors have either charge $\frac{1}{2}h_+ - \frac{1}{2}$ or $\frac{1}{2}h_+ + \frac{1}{2}$.

Their other quantum numbers are respectively,

$$\begin{aligned} Z_{a+1}^- : \quad & H_{a+1} = h + m' - m + (a+1)^2 pq + (a+1)(q - pm), \\ & (h_-)_{a+1} = \mathbf{1} + \mathbf{2}(\mathbf{a} + \mathbf{1})\mathbf{p} - m(k+1) \\ T_a^- : \quad & H_a = h + m' + a^2 pq + a(q + pm), \\ & (h_-)_a = -\mathbf{1} - \mathbf{2a}\mathbf{p} - m(k+1). \end{aligned} \quad (4.84)$$

and

$$\begin{aligned} Z_{a+1}^+ : \quad & H_{a+1} = h - m' + (a+1)^2 pq + (a+1)(q - pm), \\ & (h_-)_{a+1} = \mathbf{1} + \mathbf{2}(\mathbf{a} + \mathbf{1})\mathbf{p} - m(k+1) \\ T_a^+ : \quad & H_a = h + m - m' + a^2 pq + a(q + pm), \\ & (h_-)_a = -\mathbf{1} - \mathbf{2a}\mathbf{p} - m(k+1) \end{aligned} \quad (4.85)$$

with $a \geq 0$. The corresponding embedding diagram is given in Figure 7.

The double multiplicity of the vectors T'_{a+1} and Z'_{a+2} for $a \geq 0$ is a rather new and remarkable feature. Until recently ([17], [12]), it was common belief that the singular vectors appearing in embedding diagrams all had multiplicity one. Our analysis for $\hat{sl}(2/1; \mathbf{C})$ confirms the presence of singular vectors of higher multiplicities for particular choices of highest weight state quantum numbers (namely class IV). We can indeed generalise the MFF construction in a highly nontrivial way in order to properly take into account the complications due to the presence of nilpotent fermionic generators. Given the importance of the higher multiplicities of singular vectors, in particular in deriving character formulas, we now illustrate our technique and construct two singular vectors $T_1'^{(1)}$ and $T_1'^{(2)}$ with the quantum numbers of T_1' as they appear in (4.78).

We restrict ourselves to the case $m' < m/2$, m odd. Similar ideas can be used for m even, and/or any m' such that $m' - m \leq 0$. The vector $T_1'^{(1)}$ is easily constructed as

$$T_1'^{(1)} = w_1^{(m)} w_0^{(q-m-1)} Z'_0, \quad (4.86)$$

with the help of the generalised Weyl transformations introduced in the previous section. The second vector is far from being trivial as a descendant of the highest state Z'_0 . A reasonable starting point would be to construct the state $w_1^{(m)} Z'_0$. However, with the class IV choice of quantum numbers for Z'_0 in particular, this state identically vanishes due to its internal fermionic structure, as can be checked by using the definition (3.38). One therefore needs to ‘improve’ the state $w_1^{(m)} Z'_0$ to avoid its vanishing. As described in the following expressions, the improved state, called $\widehat{w}_1^{(m)} Z'_0$, is given by a linear combination of the appropriately dressed neutral objects $\log J_{-1}^+$ and $(J_0^-)^{(-1)} j_0^- j_0'^-$. Explicitly, the singular vector $T_1'^{(2)}$ is given by

$$T_1'^{(2)} = w_1^{(m)} w_0^{(q-m-1)} \widehat{w}_1^{(m)} Z'_0, \quad (4.87)$$

with

$$\begin{aligned} \widehat{w}_1^{(m)} Z'_0 = & 4h_+^2 \tilde{w}_0^{(m')} \left[(J_0^-)^{-h_+-1} j_0'^- j_0^- \tilde{w}_1^{(m/2-m'-1/2)} \alpha \log J_{-1}^+ \tilde{w}_0^{(m/2-m'-1/2)} (J_0^-)^{h_+-1} j_0^- j_0'^- \right. \\ & \left. + \beta (J_0^-)^{-1} j_0^- j_0'^- \right] \tilde{w}_1^{(m')} Z'_0 \end{aligned} \quad (4.88)$$

where $\alpha = h_+$ and

$$\beta = \prod_{j=1}^{m/2-m'-1/2} (h_+ + (2j-1)(k+1)) (h_+ - (2j-1)(k+1)). \quad (4.89)$$

The function $\log J_{-1}^+$ allows to write up vectors such as $T_1'^{(2)}$ in a concise way, making it easy to check they are singular. Indeed, J_1^- commutes with $\tilde{w}_0^{(M)}$ and $\tilde{w}_1^{(M)}$ for any value of M , and because the commutator

$$[J_1^-, \log J_{-1}^+] = (J_{-1}^+)^{-1} (k+1 - 2J_0^3) \quad (4.90)$$

vanishes when evaluated on a state whose quantum number H_- satisfies $k+1 - H_- = 0$ (it is always the case in our construction), one concludes that, in particular, $J_1^- T_1'^{(2)} = 0$. Although j_0^+ and $j_0'^+$ commute with $\log J_{-1}^+$, it is interesting to note that checking $j_0^+ T_1'^{(2)} = j_0'^+ T_1'^{(2)} = 0$ is not straightforward. Indeed, $\log J_{-1}^+$ is not an eigenvector of J_0^3 (although (4.88) is), i.e.,

$$[J_0^3, \log J_{-1}^+] = 1, \quad (4.91)$$

and hence the corrective term $\beta (J_0^-)^{-1} j_0^- j_0'^-$ is introduced in (4.88) in order to ensure that $j_0^+ T_1'^{(2)} = 0$.

Although $\log J_{-1}^+$ appears in the formal expression of some of our singular vectors, it does actually not survive in any evaluation of the vector, since it can be commuted through with the help of the following relations,

$$\begin{aligned} [j_0^-, \log J_{-1}^+] &= j_{-1}'^+ (J_{-1}^+)^{-1}, \\ [j_0'^-, \log J_{-1}^+] &= -j_{-1}^+ (J_{-1}^+)^{-1}, \\ [J_0^-, \log J_{-1}^+] &= -2(J_{-1}^+)^{-1} J_{-1}^3 + (J_{-1}^+)^{-2} J_{-2}^+, \end{aligned} \quad (4.92)$$

and disappears because the first term in the square bracket of (4.88) vanishes identically when $\log J_{-1}^+$ is removed.

We have concentrated here on the case where m is odd. If m is even, one substitutes appropriately the expression $[h_+ \log J_0^- + (J_0^-)^{-1} j_0^- j_0'^-]$ for $\log J_{-1}^+$.

We now proceed to indicate the relation between the fermionic singular vector T_0^+ and the two uncharged singular vectors $T_1'^{(1)}$ and $T_1'^{(2)}$. The construction given in Lemma 5 for T_0^+ , namely,

$$T_0^+ = \tilde{w}_0^{(m-m')} j_0^- \tilde{w}_1^{(m-m')} Z'_0, \quad (4.93)$$

leads to a state which vanishes identically in this class, again because of its internal fermionic nature. The actual fermionic vector is obtained by using an ‘improved’ version of Lemma 5, inspired by the idea above, namely,

$$\begin{aligned} T_0^+ = 2h_+ \tilde{w}_0^{(m-m')} \left[j_0^- \tilde{w}_1^{(m/2-m'-1/2)} \alpha \log J_{-1}^+ \tilde{w}_0^{(m/2-m'-1/2)} (J_0^-)^{h_+-1} j_0^- j_0'^- \right. \\ \left. - \beta (J_0^-)^{h_+} j_0^- \right] \tilde{w}_1^{(m')} Z'_0, \end{aligned} \quad (4.94)$$

with α and β given as before. It is now almost straightforward to identify which particular linear combination of $T_1'^{(1)}$ and $T_1'^{(2)}$ is a descendant of T_0^+ ,

$$N_1(-4h_+^2\beta N_2 T_1'^{(1)} + T_1'^{(2)}) = w_1^{(m)} w_0^{(q-m-1)} \tilde{w}_0^{(m')}(J_0^-)^{-h_+-1} (-2h_+ j_0'^-) \tilde{w}_1^{(m-m')} T_0^+, \quad (4.95)$$

where N_1 is a nonzero normalisation constant given by (3.42) for $M = m' - m$, $H_+ = -H_- = h_+ + 1$, and N_2 is similarly given by (3.43) for $M = m'$, $H_- = h_-$ and $H_+ = h_+$.

Finally, the uncharged singular vector $T_1'^{(2)}$ can be seen as a descendant of the fermionic singular vector T_0^- in the following way. By lemma 2, one constructs T_0^- as,

$$T_0^- = \tilde{w}_0^{(m')} j_0'^- \tilde{w}_1^{(m')} Z_0' \quad (4.96)$$

and

$$N' T_1'^{(2)} = 4h_+^2 \tilde{w}_0^{(m')} \left[(J_0^-)^{-h_+-1} j_0'^- j_0^- \tilde{w}_1^{(m/2-m'-1/2)} \alpha \log J_{-1}^+ \tilde{w}_0^{(m/2-m'-1/2)} (J_0^-)^{h_+-1} j_0^- + \beta (J_0^-)^{-1} j_0^- \right] \tilde{w}_1^{(m')} T_0^-. \quad (4.97)$$

The normalisation factor N' is,

$$N' = \prod_{i=0}^{m'-1} (h_+ + h_- + 2i(k+1) - 2) (h_+ - h_- - 2i(k+1)). \quad (4.98)$$

Once more, this detailed analysis illustrates very well the power of our analytic expressions in relating singular vectors between themselves within a given embedding diagram.

5 Conclusions

The Lie superalgebra $A(1,0)$ and its affinisation $A(1,0)^{(1)}$ play a crucial role in the description of noncritical $N = 2$ superstrings. In order to study the space of physical states of the latter theory, using the tool provided by topological G/G WZNW models, a detailed analysis of various modules over $A(1,0)^{(1)}$ is needed. Many Lie superalgebras share with $A(1,0)$ the property that two sets of simple roots may not be equivalent up to Weyl transformations, which are generated by reflections with respect to bosonic simple roots. An added technical complication in $A(1,0)$ is the fact that the fermionic roots are lightlike, which prevents one from defining coroots and fundamental weights in a straightforward way. These properties are emphasized in Section 2. The classical and quantum free field Wakimoto representations of $\hat{sl}(2/1; \mathbf{R})$ built with two inequivalent sets of simple roots are given in [10]. It is shown there that there exists a set of field transformations which relate the two Wakimoto representations in the classical and the quantum case. Section 3 organises the information provided by the Kac-Kazhdan determinant formula relevant to the Lie superalgebra $A(1,0)^{(1)}$ in five lemmas. The Malikov-Feigin-Fuchs construction is generalised to incorporate transformations which relate bosonic and fermionic singular vectors within a Verma module. Section 4 provides vital information for the construction of admissible representations of $A(1,0)^{(1)}$, namely the quantum numbers and embedding diagrams of the singular vectors appearing in highest weight Verma modules when the level k of the algebra satisfies the

necessary condition $k + 1 = p/q$ with p, q nonzero positive relatively prime integers. We believe that the extra conditions leading to the seven embedding diagrams of Classes I,II,III, IV are sufficient to determine all admissible representations, whose characters should provide finite representations of the modular group. Our analysis clearly shows a very close link between the embedding diagrams of Section 4 and those of some completely degenerate representations of the $N = 2$ superconformal algebra [17]. This striking similarity is reminiscent of the link between the admissible $\widehat{sl}(2; \mathbf{C})$ modules and the degenerate Virasoro modules, and between the admissible $\widehat{osp}(1/2; \mathbf{C})$ modules and the $N = 1$ degenerate $N = 1$ superconformal modules. A recent paper [38] offers some explanation of this similarity.

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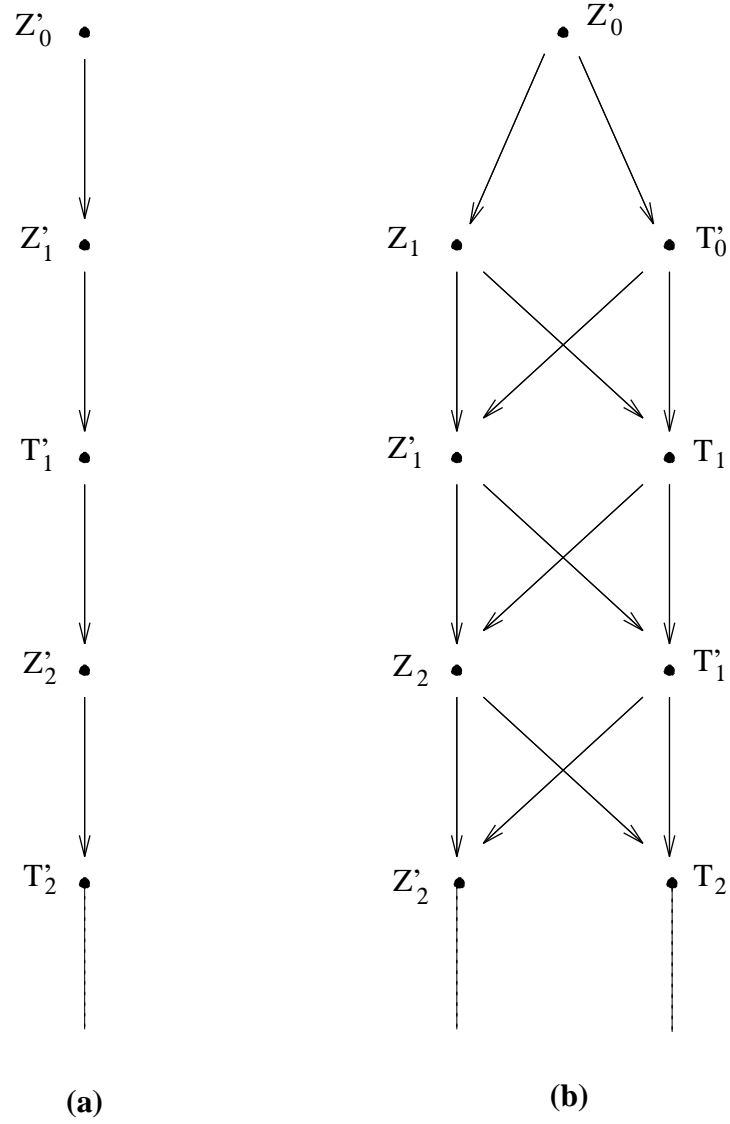


Fig 2: Class I Embedding Diagrams for (a) $n=0$ and (b) n non-zero

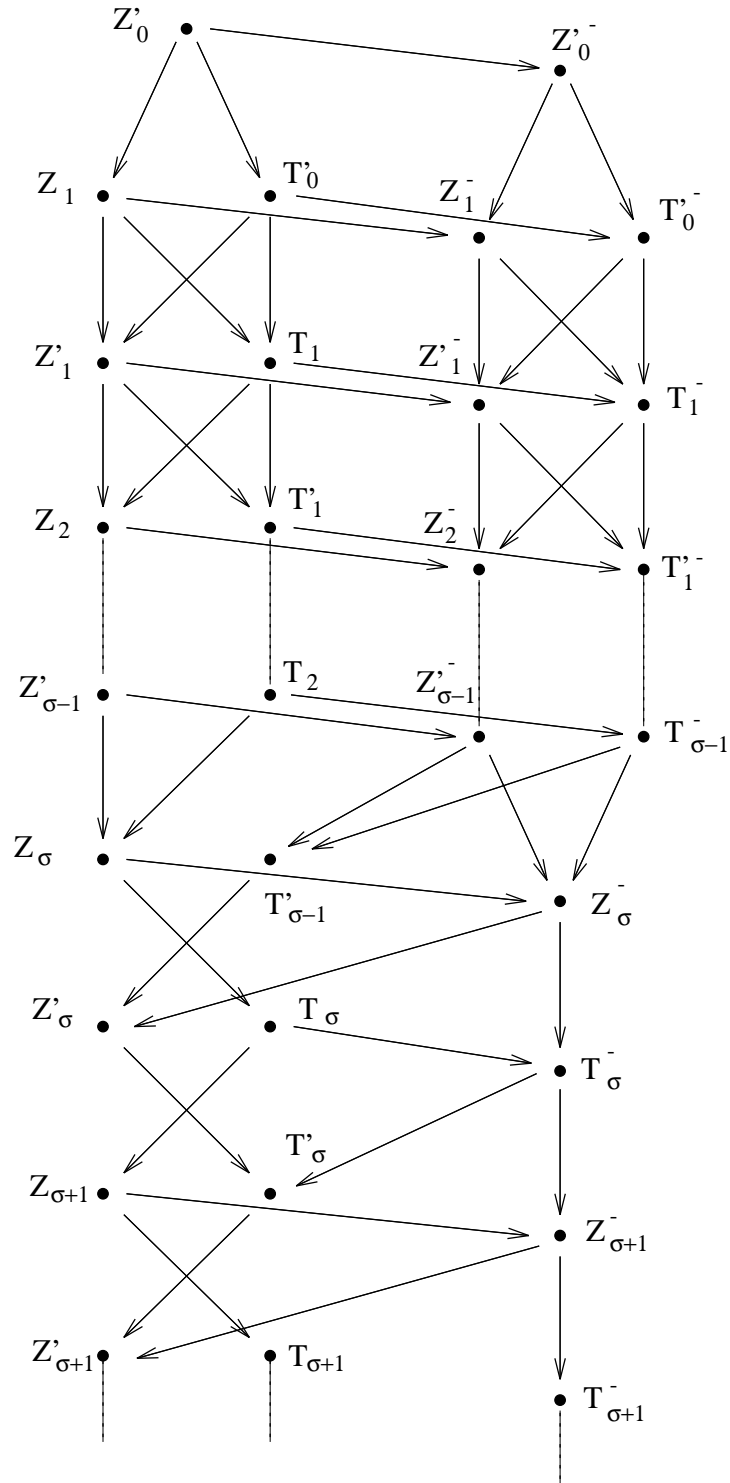
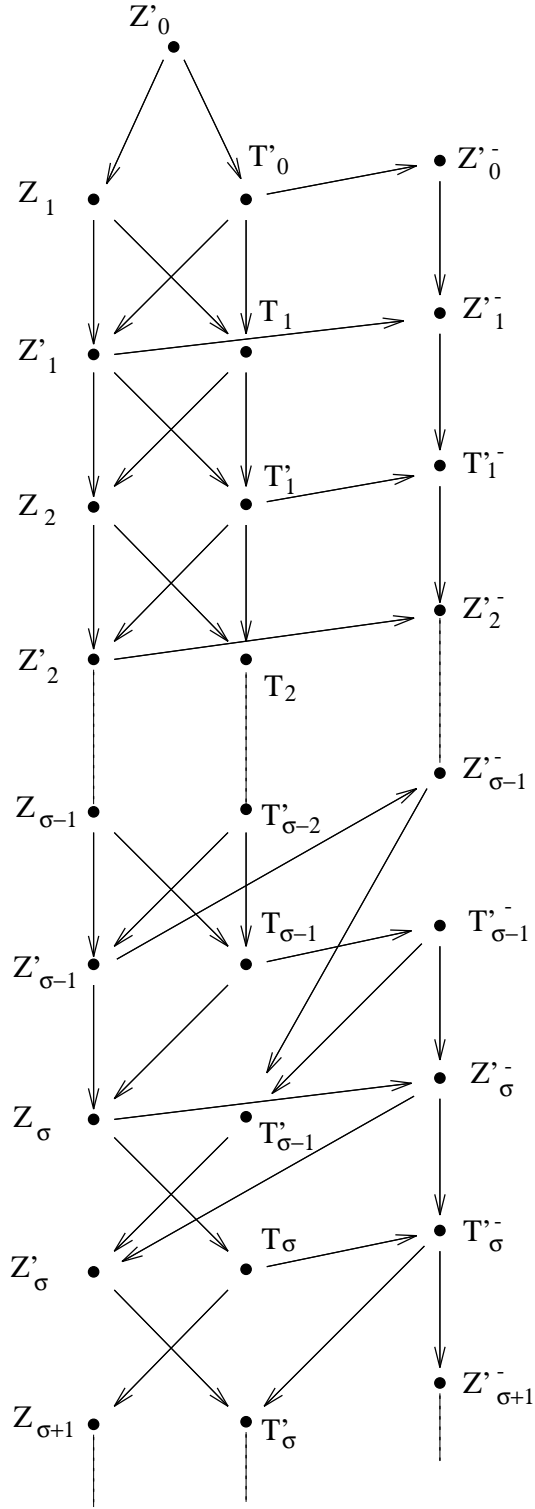
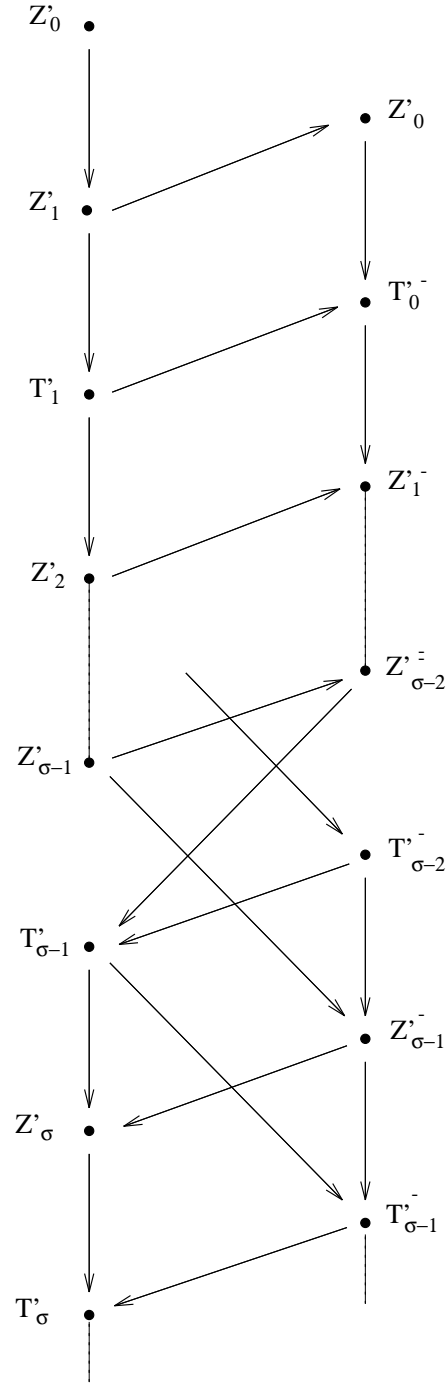
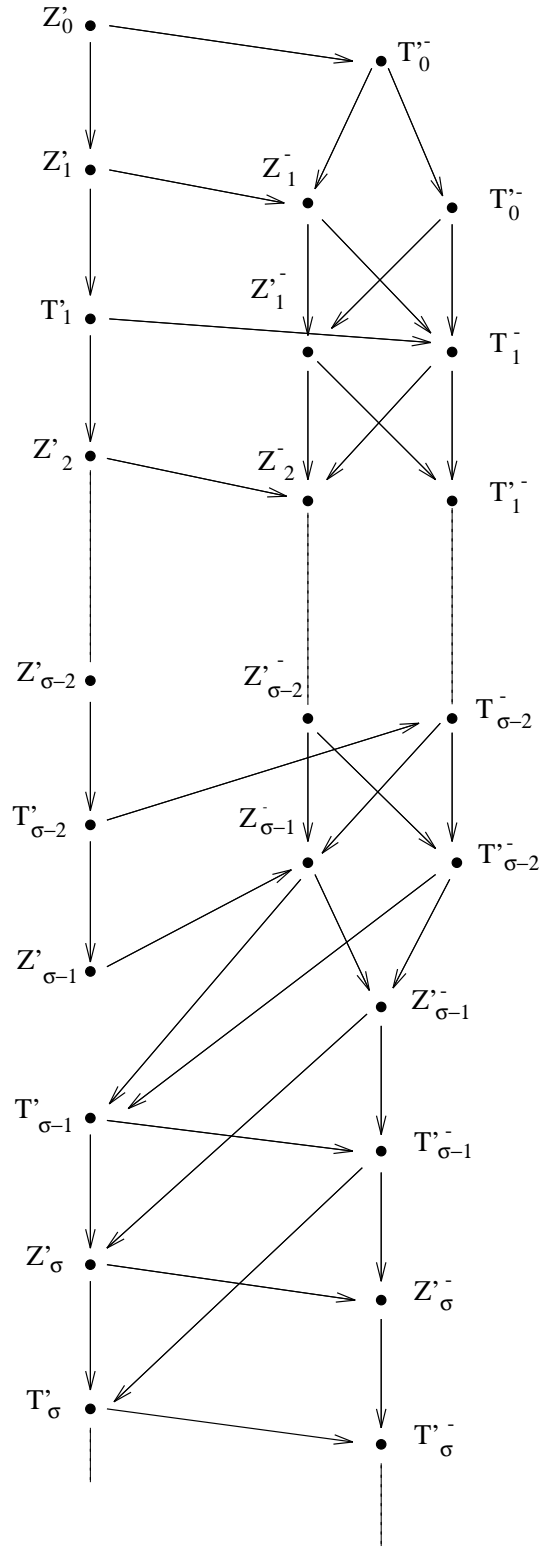


Fig 3: Class II Embedding Diagram for n other than one

Fig 4: Class II Embedding Diagram for $n=1$

Fig 5: Class III Embedding Diagram for $p=1$

Fig 6: Class III Embedding Diagram for p other than 1

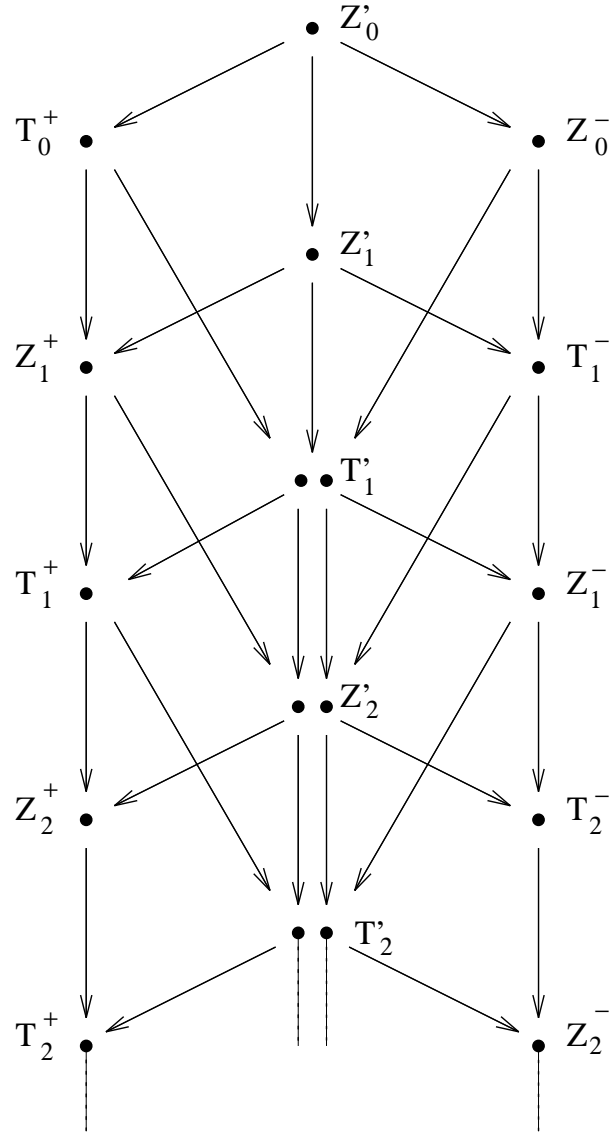


Fig 7: Class IV Embedding Diagram